Entanglement cost of antisymmetric states and additivity of capacity of some quantum channels

2004 J. Phys. A: Math. Gen. 37 L167

(http://iopscience.iop.org/0305-4470/37/15/L03)

J. Phys. A: Math. Gen. **37** (2004) L167–L171 PII: S0305-4470(04)73051-6

LETTER TO THE EDITOR

Entanglement cost of antisymmetric states and additivity of capacity of some quantum channels

Keiji Matsumoto[1,2] **and Fumitaka Yura**[2]

[1] National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda, Tokyo, 101-8430, Japan
[2] Quantum Computation and Information Project, ERATO, Japan Science and Technology Agency, Daini Hongo White Bldg 201, 5-28-3 Hongo, Bunkyo, Tokyo 113-0033, Japan

Received 5 December 2003
Published 29 March 2004
Online at stacks.iop.org/JPhysA/37/L167 (DOI: 10.1088/0305-4470/37/15/L03)

Abstract
We study the entanglement cost of the states in the antisymmetric space, which consists of $(d-1)$ d-dimensional systems. The cost is always $\log_2(d-1)$ ebits when the state is divided into bipartite $\mathbb{C}^d \otimes (\mathbb{C}^d)^{d-2}$. Combined with the arguments in [6], additivity of channel capacity of some quantum channels is also shown.

PACS numbers: 03.65.Ud, 03.67.−a, 03.67.Hk

The concept of entanglement is the key for quantum communication, quantum computing and quantum information processing. One candidate to quantify entanglement is entanglement of formation E_f. In [2], it is shown that the entanglement cost E_c to create some state can be asymptotically calculated from E_f. In this sense, the entanglement of formation E_f has an important physical meaning and is a significant quantity; the calculation of E_f is a tough problem since the definition of E_f, equation (2), contains minimization. The calculation of E_c is even harder, and in fact, E_c is computed only for several special states [1, 5, 6]. In this letter, we pay attention to antisymmetric states that are easy to deal with. In addition, Holevo capacity of quantum channels induced by antisymmetric spaces is discussed.

As for antisymmetric states, the following things are known, for example. The entanglement of formation for two states in $\mathcal{S}(\mathbb{C}^3_*)$ is additive [3], where $\mathcal{S}(\mathcal{H})$ is the set of density matrices over a Hilbert space $\mathcal{H}$ and $\mathbb{C}^3_*$ defined later. Furthermore, the lower bound to entanglement cost of density matrices in d-level antisymmetric space, obtained in [4], is $\log_2 \frac{d}{d-1}$ ebit. Recently, one of the authors showed that the entanglement cost of three-level antisymmetric states in $\mathcal{S}(\mathbb{C}^3_*)$ is exactly one ebit [1]. In this letter, we show that the entanglement cost on $\mathcal{S}(\mathbb{C}^d_*)$ is equal to $\log_2(d-1)$, which includes [1] as a special case.

Let $\mathbb{C}^d$ be $\mathrm{span}_{\mathbb{C}}\{|1\rangle, |2\rangle, \ldots, |d\rangle\}$ and $d \geqslant 3$. We first define the antisymmetric states which consist of $d-1$ particles with $SU(d)$ symmetry as follows:

$$\mathbb{C}^d_* := \mathrm{span}_{\mathbb{C}}\{|1\rangle_a, |2\rangle_a, \ldots, |d\rangle_a\} \subset \mathbb{C}^{d^{\otimes(d-1)}}$$

where $|i_1\rangle_a := \frac{1}{\sqrt{(d-1)!}} \sum_{i_2,\ldots,i_d} \epsilon_{i_1 i_2,\ldots,i_d} |i_2\rangle \otimes \cdots \otimes |i_d\rangle$, $1 \leqslant i_1, i_2, \ldots, i_d \leqslant d$ and ϵ is a totally antisymmetric tensor. When $d = 4$, for example, $|1\rangle_a = (|234\rangle - |243\rangle + |342\rangle - |324\rangle + |423\rangle - |432\rangle)/\sqrt{6}$. Suppose $U \in SU(d)$ acts on $\mathbb{C}^d$ as $U|i\rangle = \sum_j U^i_j |j\rangle$, then on $\mathbb{C}^d_*$,

$$\begin{aligned} U|i_1\rangle_a &= \frac{1}{\sqrt{(d-1)!}} \sum_{i_2,\ldots,i_d} U^{\otimes(d-1)} \epsilon_{i_1,\ldots,i_d} |i_2, \ldots, i_d\rangle \\ &= \frac{1}{\sqrt{(d-1)!}} \sum_{j_1,\ldots,j_d} (U^\dagger)^{j_1}_{i_1} \epsilon_{j_1,\ldots,j_d} |j_2, \ldots, j_d\rangle \\ &= \sum_{j_1} (U^\dagger)^{j_1}_{i_1} |j_1\rangle_a \end{aligned} \tag{1}$$

where we have used the fact that the totally antisymmetric tensors $\epsilon_{j_1,\ldots,j_d}$ are invariant under $U^{\otimes d}$. The Hermitian conjugate of U on the right-hand side suggests that $\mathbb{C}^d_*$ is the dual (contragredient) space of $\mathbb{C}^d$ [7]. The corresponding Young diagrams are

$$\mathbf{d} = \square \qquad \mathbf{d}_* = \left.\begin{array}{c} \square \\ \square \\ \vdots \\ \square \\ \square \end{array}\right\} d-1.$$

Note that the dimension of these spaces is $\dim \mathbb{C}^d = \dim \mathbb{C}^d_* = d$, though $\mathbb{C}^d_*$ is a multiparticle space. Here, let us fix the space of Alice and Bob as $\mathbb{C}^d_* = \mathcal{A} \otimes \mathcal{B}$; $\mathcal{A} := \mathbb{C}^d$, $\mathcal{B} := \mathbb{C}^{d^{\otimes(d-2)}}$, and consider the entanglement between Alice and Bob. The entanglement of formation E_f is defined as follows:

$$E_f(\rho) := \inf \sum_j p_j S(\mathrm{Tr}_{\mathcal{B}} |\psi_i\rangle\langle\psi_i|) \tag{2}$$

where p_j and $|\psi_j\rangle$ are decompositions such that $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ and the von Neumann entropy $S(\rho) := -\mathrm{Tr}\, \rho \log_2 \rho$. Let Λ_d be a 'partial trace channel', or CP map from $\mathcal{S}(\mathbb{C}^d_*)$ to $\mathcal{S}(\mathbb{C}^d)$ with $\Lambda_d(\rho) = \mathrm{Tr}_{\mathcal{B}}\, \rho$. Equation (1) implies the channel Λ_d is contravariant,

$$\Lambda_d \left(\sum_{k,l} (U^\dagger)^k_i |k\rangle_{aa} \langle l| U^l_j \right) = U \Lambda_d (|i\rangle_{aa}\langle j|) U^\dagger.$$

Furthermore, simple calculations show that

$$\Lambda_d(|i\rangle_{aa}\langle j|) = \begin{cases} \dfrac{1}{d-1}(\mathbf{1}_d - |i\rangle\langle i|) & (i = j) \\ \dfrac{-1}{d-1} |j\rangle\langle i| & (i \neq j). \end{cases} \tag{3}$$

Because $\dim \mathbb{C}^d_* = d$, for any $|\psi\rangle \in \mathbb{C}^d_*$ there exists an element U of $SU(d)$ with $|\psi\rangle = \sum_k (U^\dagger)^k_i |k\rangle_a$. Hence, due to the contravariancy of the channel Λ_d, we have

$$\begin{aligned} S(\Lambda_d(|\psi\rangle\langle\psi|)) &= S(U \Lambda_d(|i\rangle_{aa}\langle i|) U^\dagger) \\ &= S(\Lambda_d(|i\rangle_{aa}\langle i|)) = S\left(\frac{1}{d-1} (\mathbf{1}_d - |i\rangle\langle i|) \right) = \log_2(d-1). \end{aligned} \tag{4}$$

Proposition 1. *Let* $\rho \in \mathcal{S}(\mathbb{C}^d_*)$. *Then,* $E_f(\rho) = \log_2(d-1)$.

Proof. $E_f(\rho) = \inf \sum_i p_i S(\Lambda_d(|\psi_i\rangle\langle\psi_i|)) = \inf \sum_i p_i \log_2(d-1) = \log_2(d-1)$. $\square$

The subadditivity of E_f is well known [5].

$$E_f\left(\bigotimes_{i=1}^{n}\rho^{(i)}\right) \leqslant \sum_{i=1}^{n} E_f(\rho^{(i)})$$

where $\rho^{(i)}$ are density matrices on $\mathcal{A}\otimes\mathcal{B}$, i.e., bipartite states. Using proposition 1, we obtain the following:

Corollary 1. *For any* $\rho^{(i)} \in \mathcal{S}(\mathbb{C}_*^{d_i})$, $E_f\left(\otimes_{i=1}^{n}\rho^{(i)}\right) \leqslant \sum_{i=1}^{n}\log_2(d_i-1)$.

To prove the inequality of the opposite direction, we use the following lemma.

Lemma 1 (see also [1])**.** *Let X be a positive semidefinite operator such that* $\operatorname{Tr} X = 1$. *Then* $\operatorname{Tr}[-X\log X] \geqslant -\log(\operatorname{Tr} X^2)$.

Proof. Suppose $f(x) := -\log x$ over $\mathbb{R}_+$. It follows from the convexity of the function f that $f\left(\sum_i p_i x_i\right) \leqslant \sum_i p_i f(x_i)$, where $\sum_i p_i = 1$, $p_i \geqslant 0$ and $x_i > 0$. By setting $x_i = p_i (\forall i)$, we have $-\sum_i x_i \log x_i \geqslant -\log\left(\sum_i x_i^2\right)$. This inequality holds even for some x_i are equal to zero under the convention $0\log 0 = 0$. □

In the following, we denote the identity map from $\mathcal{S}(\mathcal{K})$ to $\mathcal{S}(\mathcal{K})$ by $\mathbf{I}_{\mathcal{K}}$, and $\sum |X_{ij}|^2$ by $\|X\|^2$.

Lemma 2. *For an arbitrary state* ρ *in* $\mathcal{S}(\mathcal{K}\otimes\mathbb{C}_*^d)$, *we have* $\|\mathbf{I}_{\mathcal{K}}\otimes\Lambda_d(\rho)\|^2 = \frac{1}{(d-1)^2}\{(d-2)\|\operatorname{Tr}_{\mathbb{C}_*^d}\rho\|^2 + \|\rho\|^2\}$. *Here, the dimension of* $\mathcal{K}$ *is arbitrary.*

Proof. Decompose $\rho \in \mathcal{S}(\mathcal{K}\otimes\mathbb{C}_*^d)$ into the sum $\sum_{i,j} |i\rangle_{aa}\langle j| \otimes \rho_{ij}$, where ρ_{ij} are operators in $\mathcal{K}$. Due to equations (3), we have

$$\begin{aligned}\|\mathbf{I}_{\mathcal{K}}\otimes\Lambda_d(\rho)\|^2 &= \left\| \frac{1}{d-1}\sum_i\sum_{j\neq i} |i\rangle\langle i|\otimes\rho_{jj} - \frac{1}{d-1}\sum_{i,j\neq i}|i\rangle\langle j|\otimes\rho_{ji}\right\|^2 \\ &= \frac{1}{(d-1)^2}\left\{\sum_k\left\|\sum_{i\neq k}\rho_{ii}\right\|^2 + \sum_{i\neq j}\|\rho_{ij}\|^2\right\}.\end{aligned}$$

The first term of the last part of the equation is rewritten as follows:

$$\begin{aligned}\sum_k\left\|\sum_{i\neq k}\rho_{ii}\right\|^2 &= \sum_k\sum_{i\neq k, j\neq k}\operatorname{Tr}\rho_{ii}\rho_{jj} = (d-1)\sum_i\|\rho_{ii}\|^2 + (d-2)\sum_{i\neq j}\operatorname{Tr}\rho_{ii}\rho_{jj} \\ &= (d-2)\left\|\sum_i\rho_{ii}\right\|^2 + \sum_i\|\rho_{ii}\|^2.\end{aligned}$$

Hence, after all we have

$$\begin{aligned}\|\mathbf{I}_{\mathcal{K}}\otimes\Lambda_d(\rho)\|^2 &= \frac{1}{(d-1)^2}\left\{(d-2)\left\|\sum_i\rho_{ii}\right\|^2 + \sum_{i,j}\|\rho_{ij}\|^2\right\} \\ &= \frac{1}{(d-1)^2}\left\{(d-2)\|\operatorname{Tr}_{\mathbb{C}_*^d}\rho\|^2 + \|\rho\|^2\right\}\end{aligned}$$

and the lemma is proven. □

Lemma 3. *For any* $\rho \in \mathcal{S}\left(\mathcal{K} \otimes \bigotimes_{i=1}^{n} \mathbb{C}_*^{d_i}\right)$, $\left\|\mathbf{I}_{\mathcal{K}} \otimes \bigotimes_{i=1}^{n} \Lambda_{d_i}(\rho)\right\|^2 \leqslant \prod_{i=1}^{n} \frac{1}{d_i - 1}$, *where the dimension of* $\mathcal{K}$ *is arbitrary.*

Proof. Induction is used for the proof. First, for $n = 1$, the assertion follows directly from lemma 2, because $\|\sigma\| \leqslant 1$ holds for any density matrix σ. Second, let us assume the assertion is true for $n - 1$. Then, lemma 2 implies

$$\begin{aligned} \left\|\mathbf{I}_{\mathcal{K}} \otimes \bigotimes_{i=1}^{n} \Lambda_{d_i}(\rho)\right\|^2 &= \frac{1}{(d_n - 1)^2} \left\{ (d_n - 2) \left\|\mathbf{I}_{\mathcal{K}} \otimes \bigotimes_{i=1}^{n-1} \Lambda_{d_i}(\mathrm{Tr}_{\mathbb{C}_*^{d_n}} \rho)\right\|^2 \right. \\ &\quad \left. + \left\|\mathbf{I}_{\mathcal{K} \otimes \mathbb{C}_*^{d_n}} \otimes \bigotimes_{i=1}^{n-1} \Lambda_{d_i}(\rho)\right\|^2 \right\} \\ &\leqslant \frac{1}{(d_n - 1)^2} \left\{ (d_n - 2) \prod_{i=1}^{n-1} \frac{1}{d_i - 1} + \prod_{i=1}^{n-1} \frac{1}{d_i - 1} \right\} = \prod_{i=1}^{n} \frac{1}{d_i - 1} \end{aligned}$$

where the inequality in the second line comes from the assumption of induction. Thus, the lemma is proven. □

The following lemma is a bit weaker version of 'strong superadditivity' [8]. Hereafter, the reduced density matrix $\mathrm{Tr}_{\mathbb{C}_*^{d_1} \otimes \cdots \otimes \mathbb{C}_*^{d_{i-1}} \otimes \mathbb{C}_*^{d_{i+1}} \otimes \cdots \otimes \mathbb{C}_*^{d_n}} \rho$ is denoted by $\rho|_{\mathbb{C}_*^{d_i}}$.

Proposition 2. *For any* $\rho \in \mathcal{S}\left(\bigotimes_{i=1}^{n} \mathbb{C}_*^{d_i}\right)$, $E_f(\rho) \geqslant \sum_{i=1}^{n} \log_2(d_i - 1) = \sum_{i=1}^{n} E_f(\rho|_{\mathbb{C}_*^{d_i}})$.

Proof.

$$\begin{aligned} E_f(\rho) &= \inf \sum_i p_i S\left(\bigotimes_{j=1}^{n} \Lambda_{d_j}(|\psi_j\rangle\langle\psi_j|) \right) \geqslant -\inf \sum_i p_i \log_2 \left\| \bigotimes_{j=1}^{n} \Lambda_{d_j}(|\psi_i\rangle\langle\psi_i|) \right\|^2 \\ &\geqslant -\inf \sum_i p_i \log_2 \prod_{j=1}^{n} \frac{1}{d_j - 1} = \sum_{j=1}^{n} \log_2(d_j - 1). \end{aligned}$$

The first and the second inequalities come from lemmas 1 and 3, respectively. □

Theorem 1. *For any* $\rho^{(i)} \in \mathcal{S}(\mathbb{C}_*^{d_i})$, E_f *is additive,* $E_f\left(\otimes_{i=1}^{n} \rho^{(i)}\right) = \sum_{i=1}^{n} \log_2(d_i - 1) = \sum_{i=1}^{n} E_f(\rho^{(i)})$.

Proof. From corollary 1 and proposition 2, this theorem holds. □

As a corollary of this theorem, we finally obtain the first main result:

Corollary 2 (Main result (1))**.** $E_f(\rho^{\otimes n}) = n \log_2(d - 1)$ *for any* $\rho \in \mathcal{S}(\mathbb{C}_*^d)$. *Therefore, we obtain*

$$E_c(\rho) := \lim_{n \to \infty} \frac{1}{n} E_f(\rho^{\otimes n}) = \log_2(d - 1).$$

E_f and Holevo capacity $C(\Lambda_d)$ are related to each other [6],

$$C(\Lambda_d) := \sup_{\{p_i, \rho_i\}} \left\{ S\left(\Lambda_d \left(\sum_i p_i \rho_i \right) \right) - \sum_i p_i S(\Lambda_d(\rho_i)) \right\} = \sup_{\rho \in \mathcal{S}(\mathbb{C}_*^d)} \{S(\rho) - E_f(\rho)\}.$$

Combined with proposition 1, we have,

$$C(\Lambda_d) = \sup_{\rho\in\mathcal{S}(\mathbb{C}_*^d)} S(\rho) - \log_2(d-1) = \log_2 \frac{d}{d-1}.$$

The following corollary, which is our second main result, is derived from proposition 2 using almost the same argument as in section 5 of [6].

Corollary 3 (Main result (2))**.** *Quantum channels* Λ_{d_i} *are additive,* $C\left(\bigotimes_{i=1}^n \Lambda_{d_i}\right) = \sum_{i=1}^n C\left(\Lambda_{d_i}\right) = \sum_{i=1}^n \log_2 \frac{d_i}{d_i-1}$.

Proof.

$$C\left(\bigotimes_{i=1}^n \Lambda_{d_i}\right) = \sup_{\rho\in S(\bigotimes_{i=1}^n \mathbb{C}_*^{d_i})} \{S(\rho) - E_f(\rho)\} \leqslant \sup_{\rho\in S(\bigotimes_{i=1}^n \mathbb{C}_*^{d_i})} \left\{S(\rho) - \sum_{i=1}^n E_f(\rho|_{\mathbb{C}_*^{d_i}})\right\}$$

$$\leqslant \sup_{\rho\in S(\bigotimes_{i=1}^n \mathbb{C}_*^{d_i})} \sum_{i=1}^n \{S(\rho|_{\mathbb{C}_*^{d_i}}) - E_f(\rho|_{\mathbb{C}_*^{d_i}})\} \leqslant \sum_{i=1}^n C\left(\Lambda_{d_i}\right).$$

Here, the first inequality comes from 'strong superadditivity', proposition 2, and the second inequality is due to the superadditivity of joint entropy, $S(\rho) \leqslant \sum_{i=1}^n S(\rho|_{\mathbb{C}_*^{d_i}})$. Combined with the well-known superadditivity of Holevo capacity $C\left(\bigotimes_{i=1}^n \Lambda_{d_i}\right) \geqslant \sum_{i=1}^n C\left(\Lambda_{d_i}\right)$, the assertion is proven. □

Acknowledgments

The authors are grateful to H Fan and T Shimono for useful discussions and to Professor H Imai for support and encouragement.

References

[1] Yura F 2003 *J. Phys. A: Math. Gen.* **36** L237
[2] Hayden P M, Horodecki M and Terhal B M 2001 *J. Phys. A: Math. Gen.* **34** 6891
[3] Shimono T 2003 *Preprint* quant-ph/0301011
[4] Shimono T 2002 *Preprint* quant-ph/0203039
[5] Vidal G, Dür W and Cirac J I 2002 *Phys. Rev. Lett.* **89** 27901
[6] Matsumoto K, Shimono T and Winter A 2002 *Preprint* quant-ph/0206148 (*Commun. Math. Phys.* at press)
[7] Goodman R and Wallach N R 1998 *Representations and Invariants of the Classical Groups* (Cambridge: Cambridge University Press)
[8] Vollbrecht K G H and Werner R F 2001 *Phys. Rev.* A **64** 062307